Combining Widening and Acceleration in Linear Relation Analysis

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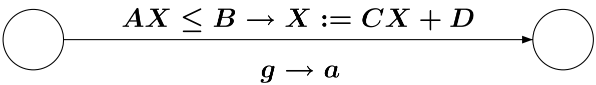
1. Introduction and Motivation
2. Simple loops
3. Two translation loops
4. Translations and reset loops
5. Implementation - ASPIC

Introduction

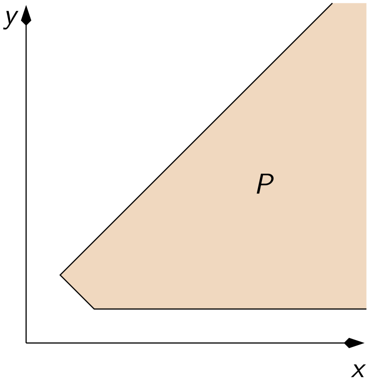
Context : Linear Relation Analysis (LRA) [Cousot&Halbwachs 78]
The analysis framework

- CFG with numerical variables and **affine** guards and actions :



- Forward and/or backward analysis to compute invariants.
- Polyhedral (over-) approximations of invariants sets. (Convex polyhedra in $\mathcal{P}(\mathbb{Q}^n)$ in **double description**).
- Possibly infinite sequences $\Rightarrow$ **widening operator**.

The double description



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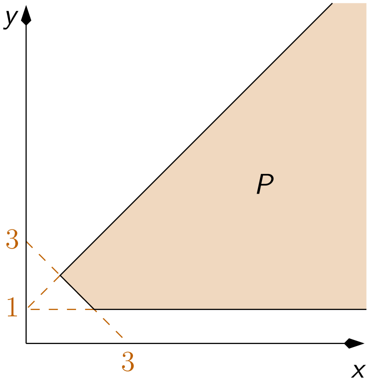

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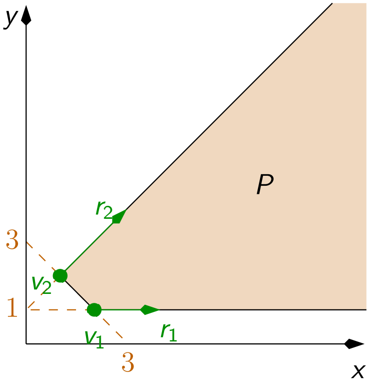

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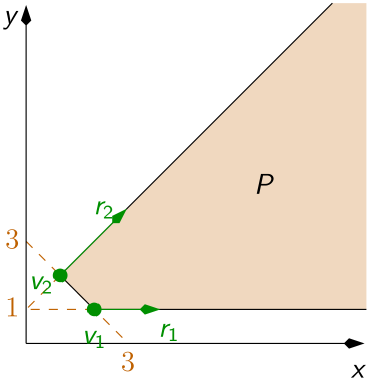

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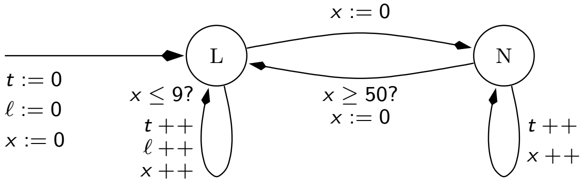
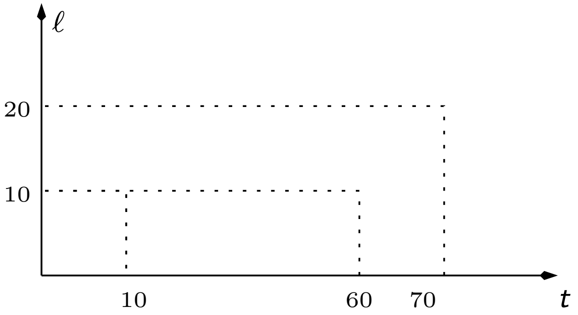
Adding rays operator

$$P \nearrow R = \{x + \sum_{r_j \in R} \mu_j r_j \mid x \in P, \mu_j \in \mathbb{Q}^+\}$$

Example - Gaz Burner - 1





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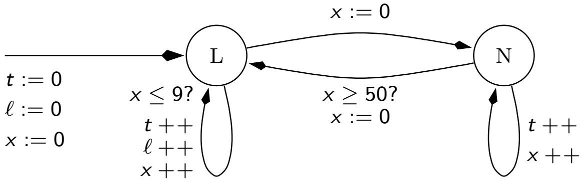

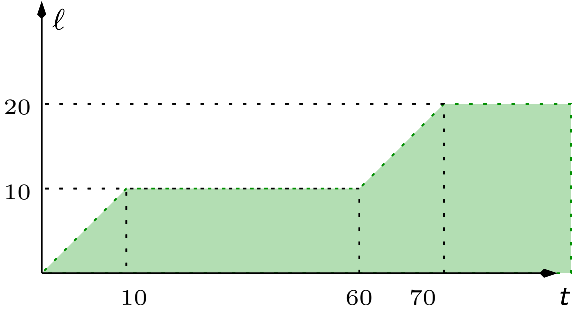

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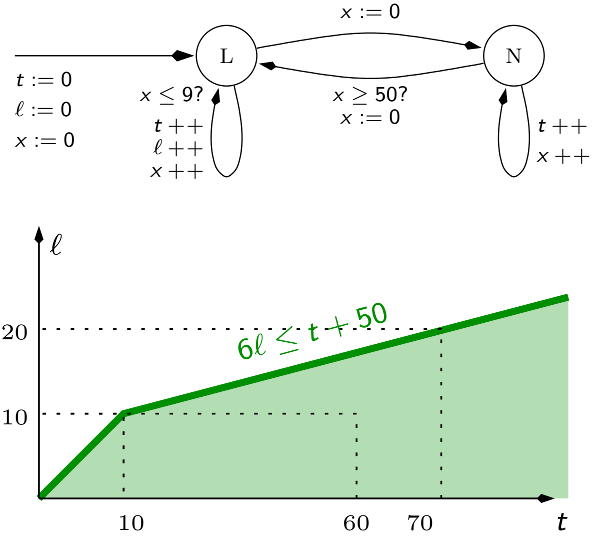
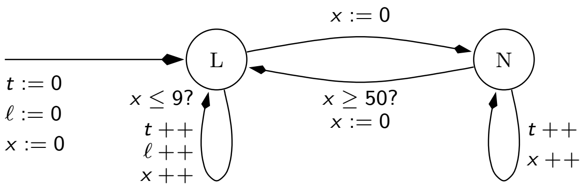


Example - Gaz Burner - 2



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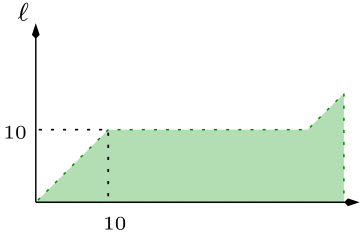

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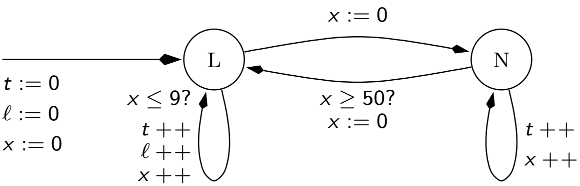

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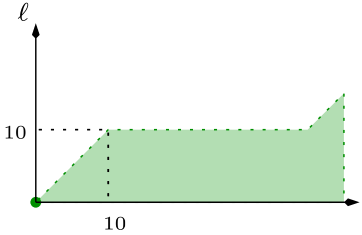

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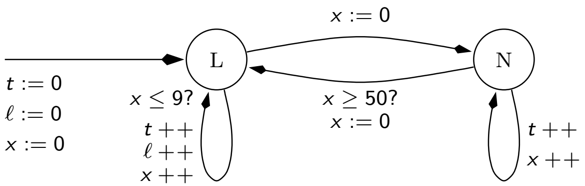

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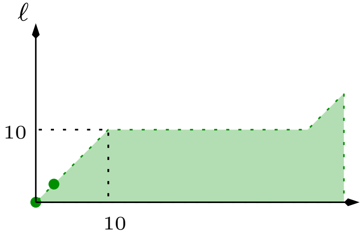

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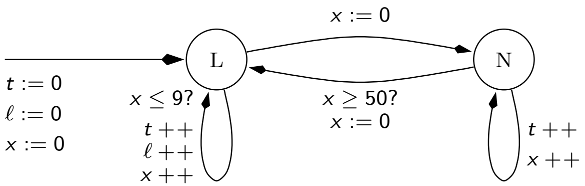

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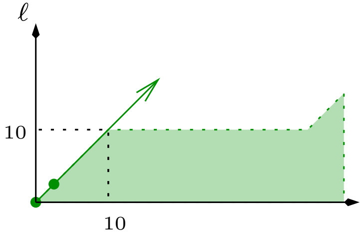

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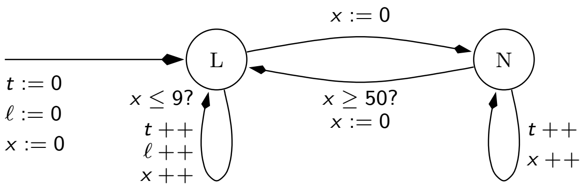

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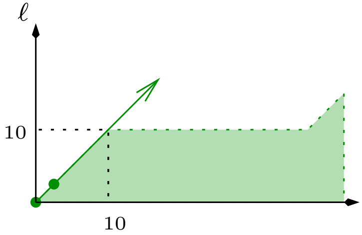

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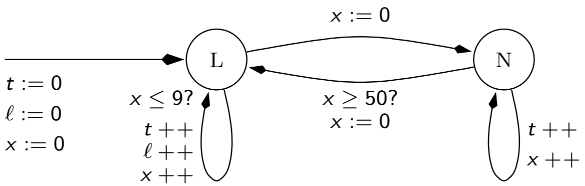

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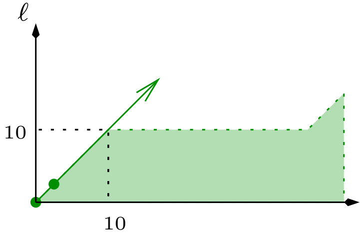

- **lack of precision** : descending sequence will not recover
- **delaying ?** 60 times !

Existing techniques (1)

Linear Relation Analysis (LRA) and Widening

- Approximate results
- Possible improvements :
  - **Delaying** the application of the widening, drawback : the cost.
  - **Changing** the operator. [Bagnara&Hill&Zafanella], drawback : no result on the global convergence.
  - Alternating widening and narrowing [Gopan&Reps : CAV 2006], drawback : the cost.

Existing techniques (2)

Acceleration

[Boigelot&Wolper, Common&Jurski, Finkel&Sutre&Leroux]

- Computing the **exact effect** of loops on integer sets.
- Encoding : Automata representing Presburger Formula
- **Drawbacks** : restricted class of programs, semi-algorithms, high complexity

▶ Our global objective : "abstract" acceleration **at low cost** for convex polyhedra combined with widening.

▶ More precisely : compute the convex hull of the exact reachable states.

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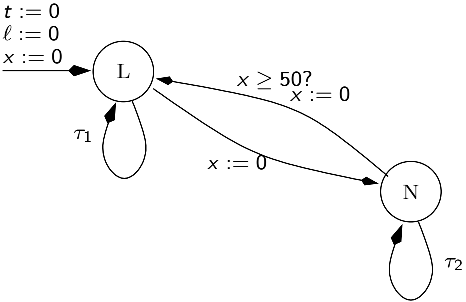
Acceleration within LRA

We want to replace the loops ($\tau_i : g_i \longrightarrow a_i$)



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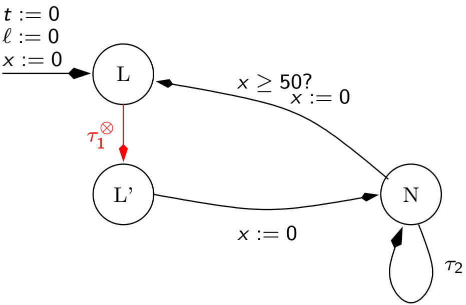

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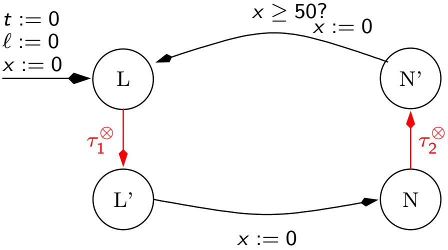

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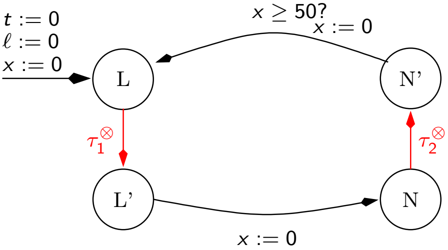

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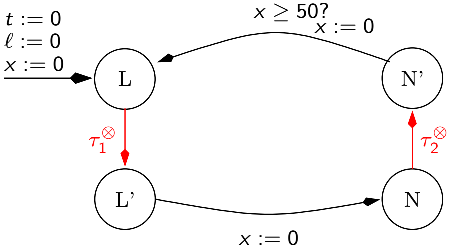

- $\tau_i^{\otimes}$ summarizes the effect of **any** number of applications of τ_i
- global loop : **accelerate** or **widen**

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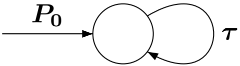
Simple loops

We want to **characterise** $\tau^*(P_0) = \bigcup_{i \in \mathbb{N}} \tau^i(P_0)$, with :

$\tau(x) =$ if $Ax \leq B$ then $Cx + D$ else x



Previous result : [Boigelot99,Leroux02] : if $\exists p, C^{2p} = C^p$ and φ a Presburger set, then $\tau^*(\varphi)$ is a computable Presburger set.

New result (not in the SAS paper) If $\exists p, C^{2p} = C^p$, then computing $\tau^*(P_0)$ reduces itself to computing $\tau'^*(P'_0)$:

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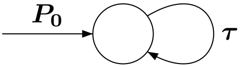

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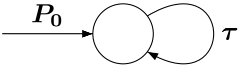

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Single translation loop

$$\tau(x) = \text{ if } Ax \leq B \text{ then } x + D \text{ else } x$$

- (obvious) Proposition

$$\tau^*(P_0) = \{x \mid \exists i \in \mathbb{N}, \exists x_0 \in P_0, \\ Ax_0 \leq B, A(x - D) \leq B, x = x_0 + iD\} \cup P_0$$

Not convex

- Discrete vs Dense acceleration

$$\tau^{\otimes}(P_0) = \{x \mid \exists i \in \mathbb{Q}^+, \exists x_0 \in P_0, \\ Ax_0 \leq B, A(x - D) \leq B, x = x_0 + iD\} \sqcup P_0$$

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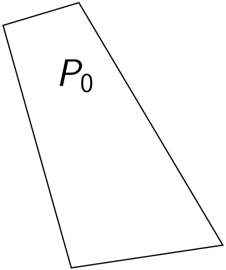
Single translation loop (2)

$$\tau(x) = \text{ if } Ax \leq B \text{ then } x + D \text{ else } x$$

Computation : Adding rays :

$$\tau^{\otimes}(P_0) = ((P_0 \cap (Ax \leq B)) \nearrow \{D\}) \cap (A(x - D) \leq B) \sqcup P_0$$

where $P \nearrow \{D\} = \{X + iD \mid X \in P, i \in \mathbb{Q}^+\}$



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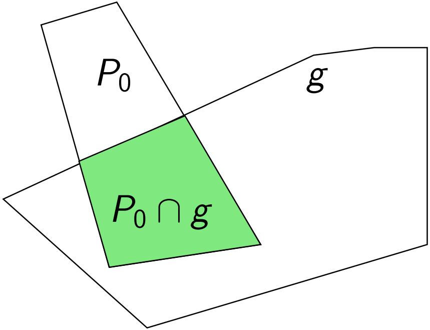

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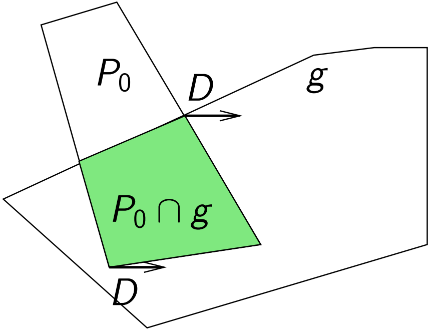

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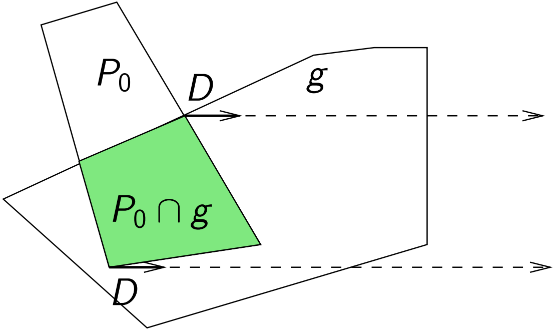

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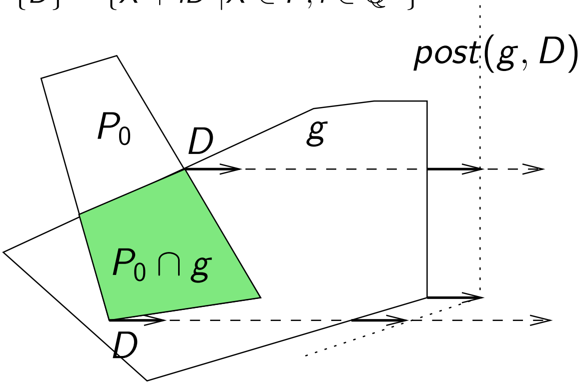

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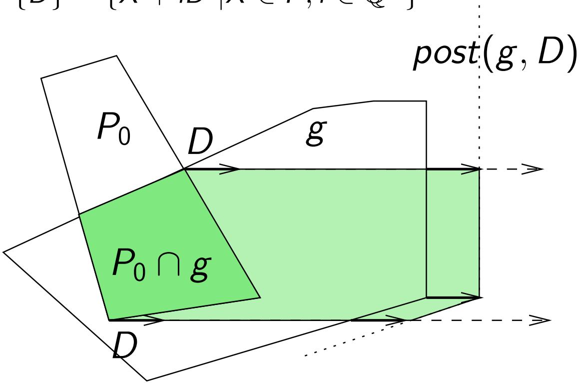

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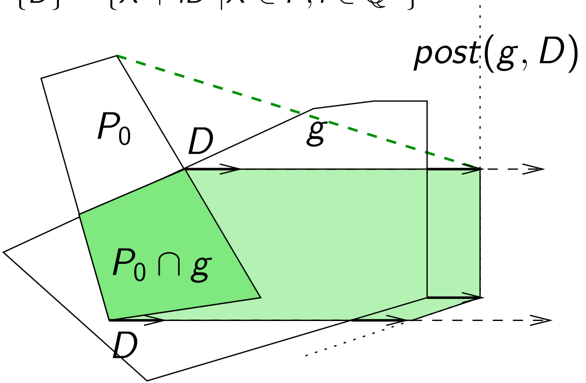

Finite Monoide simple loops

What do we lose ?

$$\tau = \begin{cases} g = (x \leq 7) \\ a = (x := x + 2) \end{cases}$$

$$P_0 = \{x = 0\}$$

$$\tau^*(P_0) = \{0 \leq x \leq 8\}$$

$$\begin{aligned} \tau^{\otimes}(P_0) &= P_0 \nearrow (1) \cap (x - 2 \leq 7) \\ &= \{0 \leq x \leq 9\} \end{aligned}$$

New result (not in the SAS paper) If $\exists p, C^p = C^{2p}$, we can compute in 2-exp an over approximation of $\tau^*(P_0)$. We note it $\tau^{\otimes}(P_0)$.

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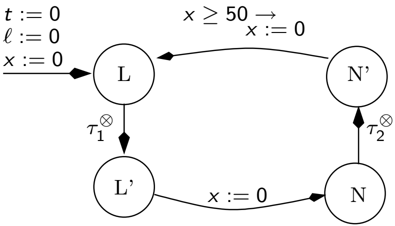

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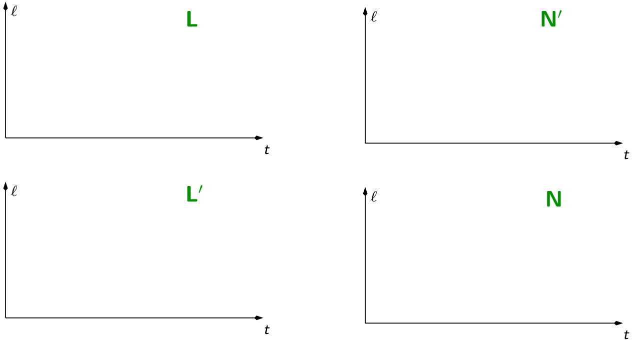
- $\tau_2^{\otimes}$ = "add-ray $(1, 0, 1)$"



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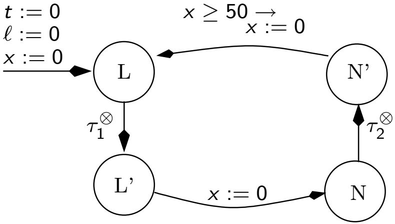

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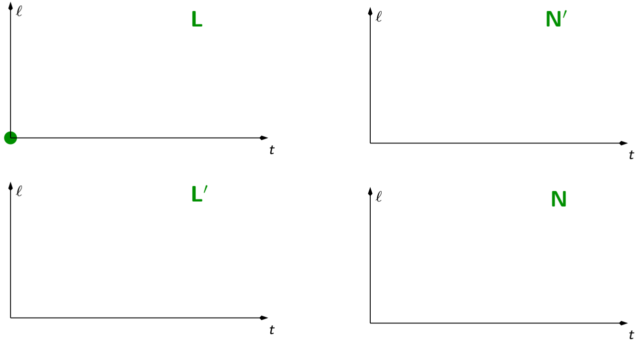

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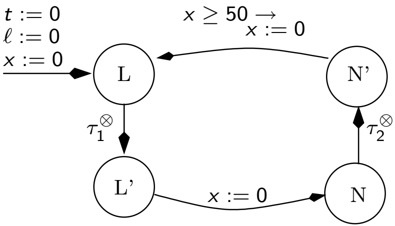

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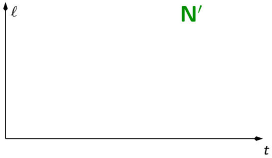

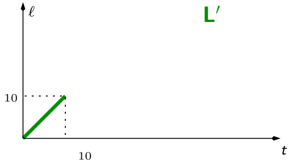

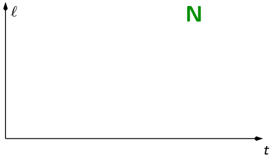

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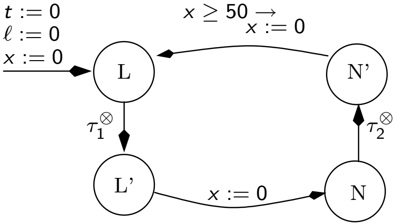

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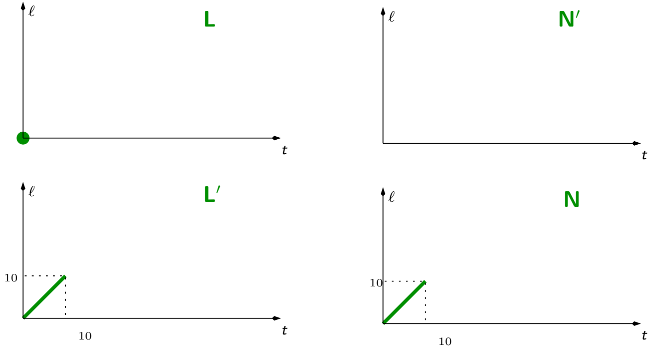

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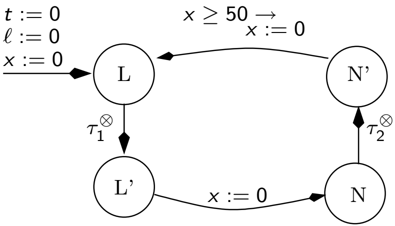

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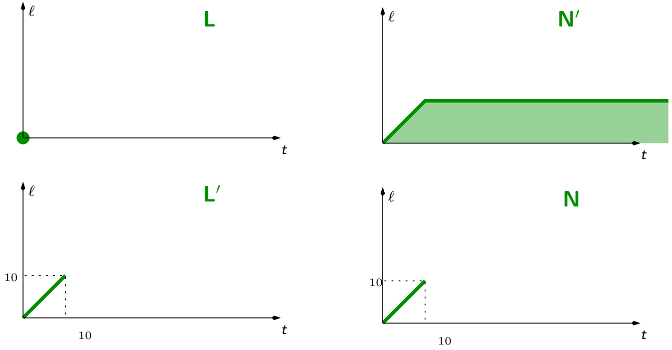

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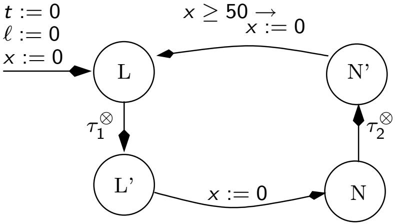

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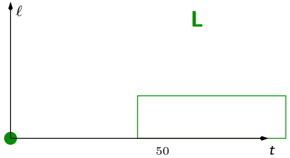

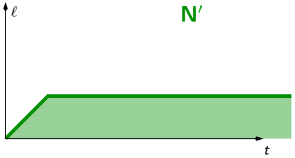

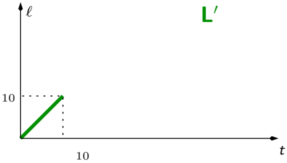

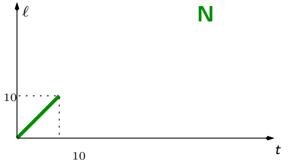

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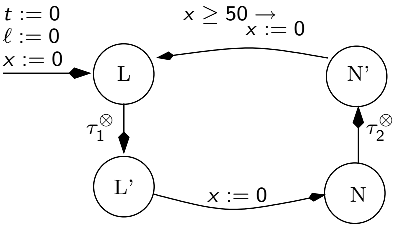

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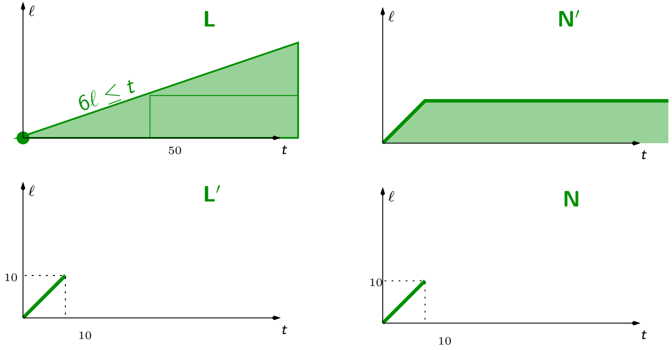

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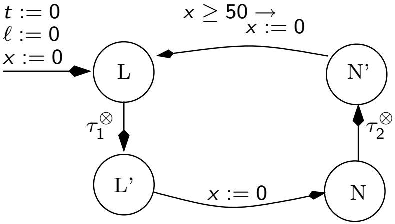

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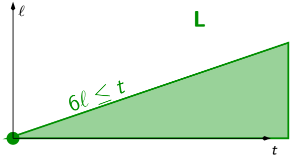

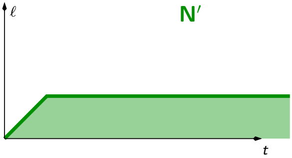

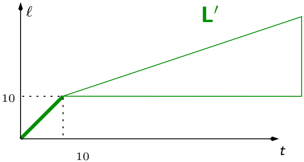

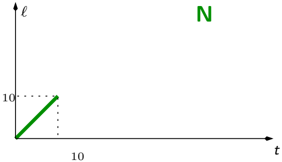

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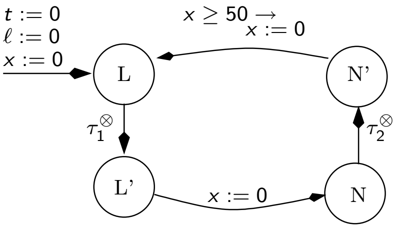

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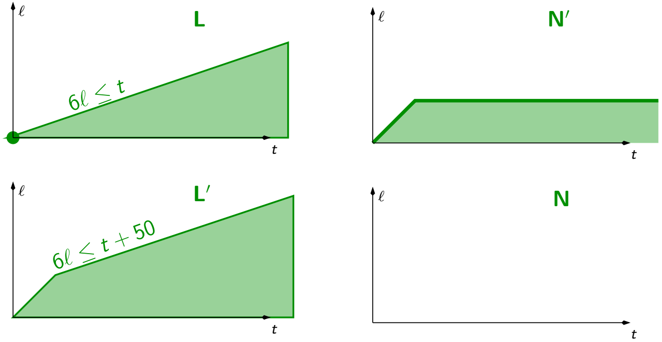

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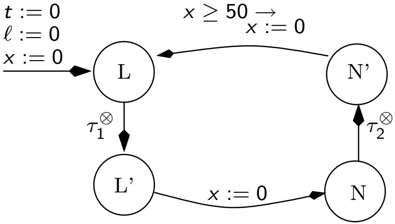

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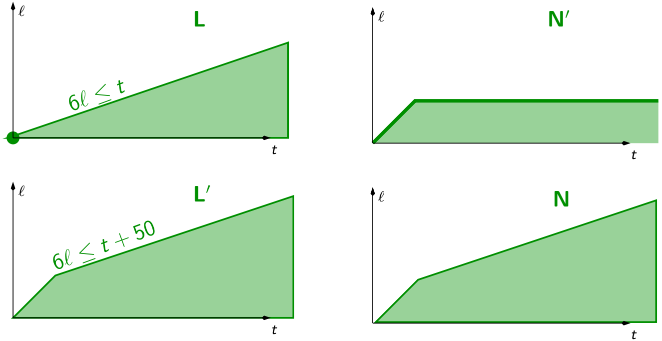

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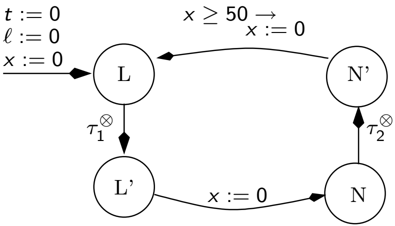

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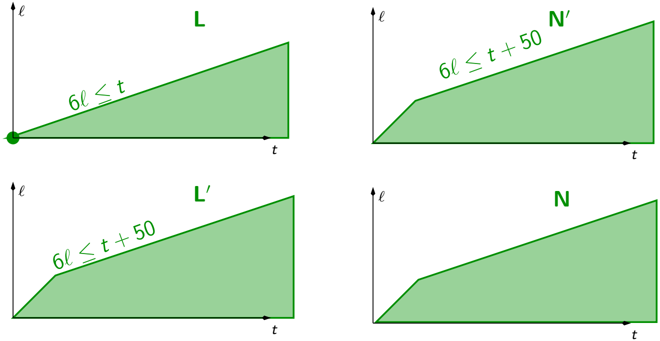
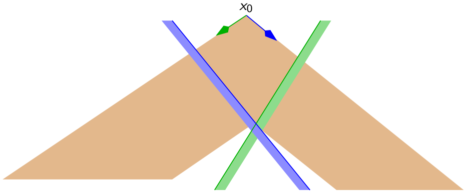


1. Introduction and Motivation
2. Simple loops
3. Two translation loops
4. Translations and reset loops
5. Implementation - ASPIC

Two loops - First remarks

$(\tau_1 + \tau_2)^*(P_0)$ is not necessarily **convex** :



Two loops - First remarks

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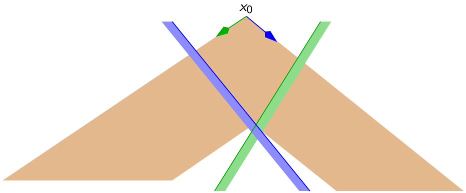
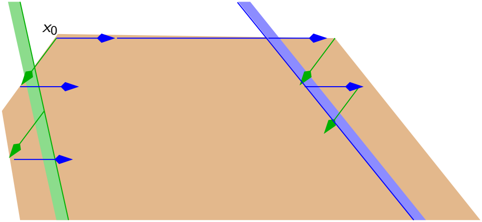
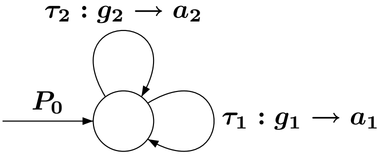


There can be quite complex **oscillations**



Two simple translation loops – new results



- New result for $g_i = G_i X_i \leq c_i$: an algorithm to compute an over-approximation of $(\tau_1 + \tau_2)^*(P_0)$ (convex polyhedron).
- In other cases, add a new loop computing P_1 :

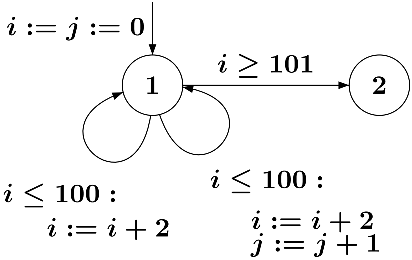
$$P_1 = P_0 \sqcup \tau_1^{\otimes}(P') \sqcup \tau_2^{\otimes}(P')$$

where :

$$P' = (P_0 \cap g_1 \cap g_2 \nearrow \{D_1, D_2\}) \cap g_1 \cap g_2$$

Old Cousot&Halbwachs78 example

```
  i :=0;
  j :=0;
  while i<= 100 do
1     if ? then i :=i+2
      else i :=i+2; j :=j+1
      fi
  od
2
```



Immediate result at control point 1

$$(0,0) \nearrow \{(2,0),(2,1)\} \cap (i \leq 100)$$
$$= 0 \leq 2j \leq i \leq 100$$

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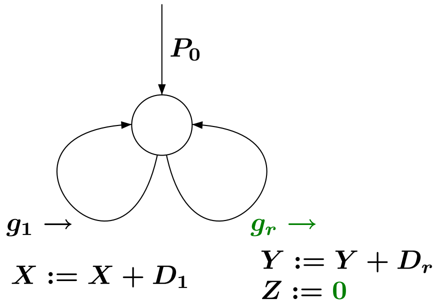
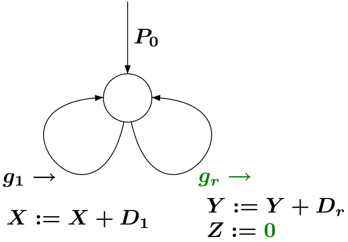
Two loops with reset (or constant assignments)

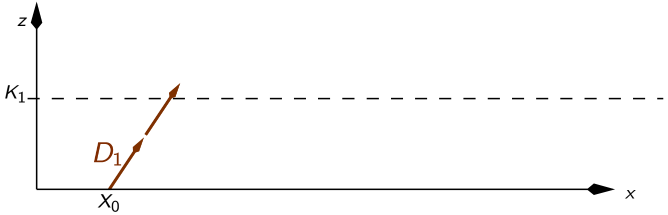


We suppose $P_0 \subseteq \{Z = 0\}$

Unconditional simple translation/reset (2)



$g_1 : Z \leq K_1,\ g_r = true$



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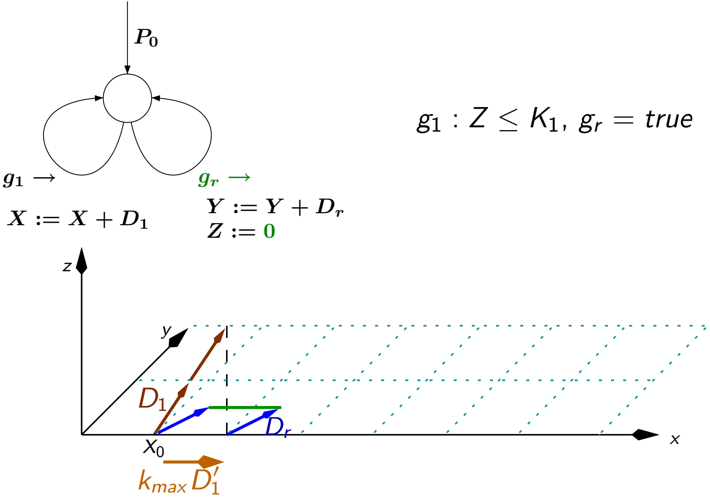

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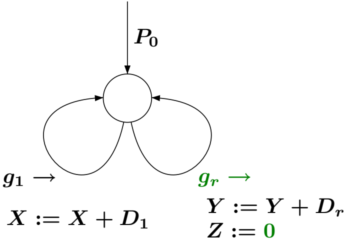

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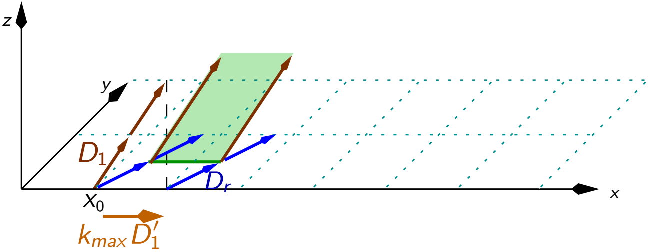

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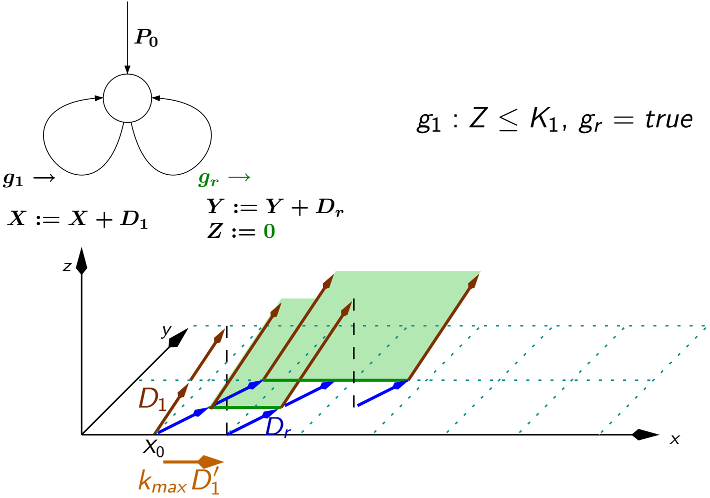

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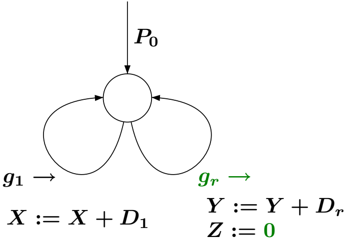

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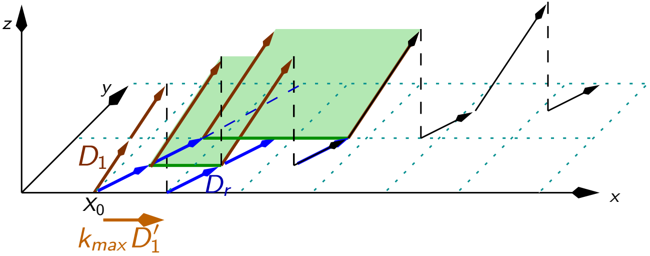

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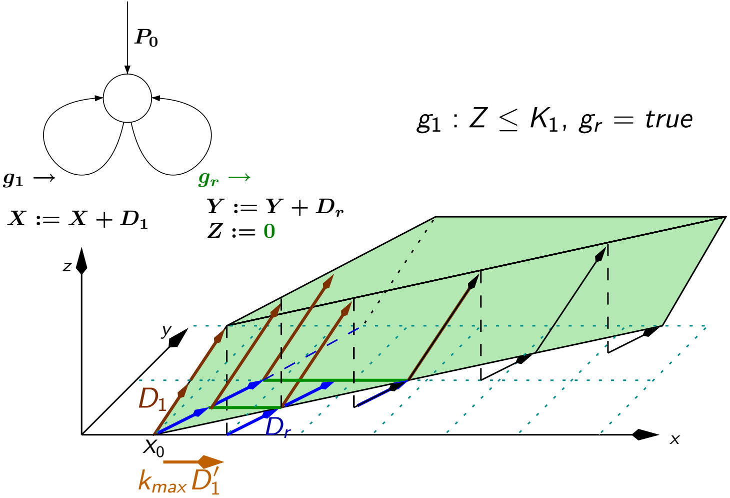

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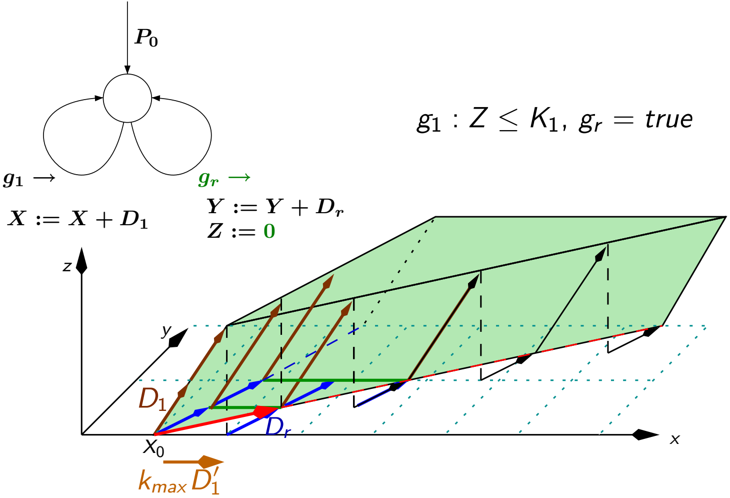

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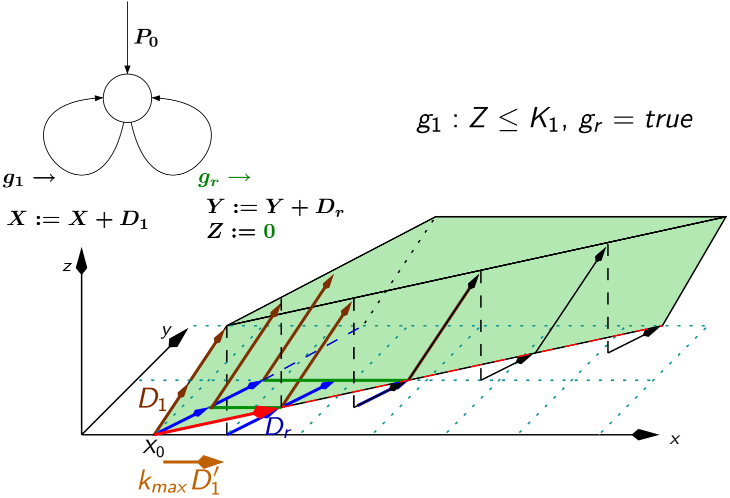
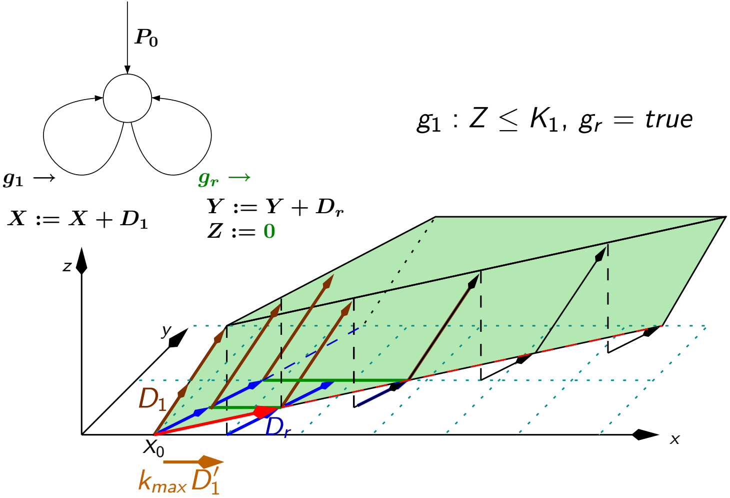


$$P_0 \nearrow \{D_1, D_r, k_{max} D_1' + D_r\} \cap post(g_1)$$

Unconditional simple translation/reset (2)

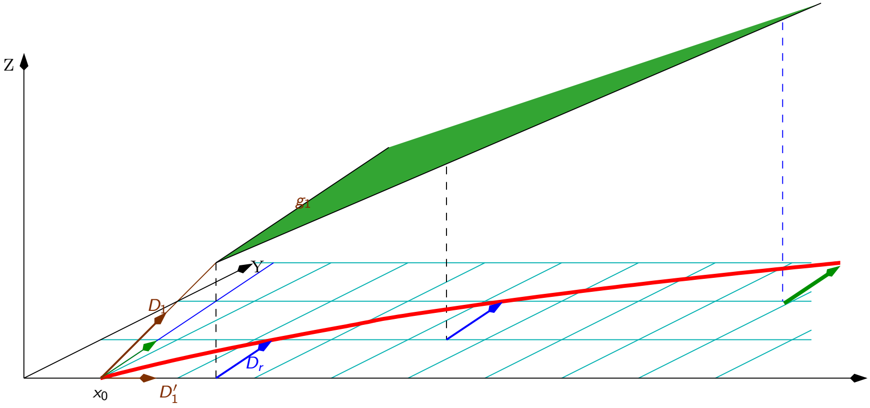


$$P_0 \nearrow \{D_1, D_r, k_{max} D_1' + D_r\} \cap post(g_1)$$

- More cases in the SAS paper.

Unconditional simple translation/reset (3)

But ... Only works if $g_1 = Z \leq K_1$ (modulo variable change)



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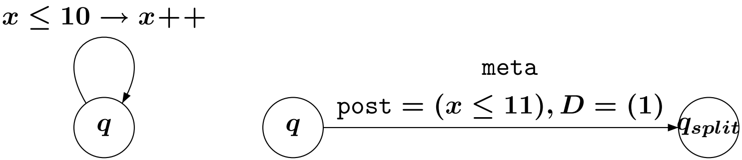
Overview of ASPIC

- A Fixpoint engine for the polyhedral lattice [B. Jeannet]
- Creation of **meta-transitions** if possible :



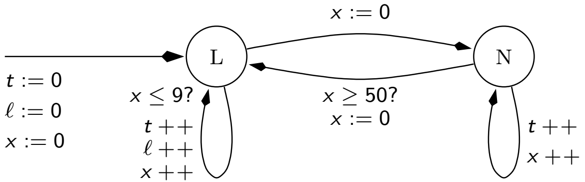
- Classical LRA with forward computation, widening and **meta-transitions**.

Aspic demo

ASPIC : Accelerated Symbolic Polyhedral Invariant Computation

http://www-verimag.imag.fr/~gonnord/aspic/aspic.html

demo.




```
Region bad := {6l>t+50}
```

Currently working on

- More general cases of loops.
- Lustre input.
- Sets of benchmarks coming from :
  - Sensor networks : consumption properties (Maraninchi/Samper)
  - "Programs with list are counter automata" (CAV 06, Iosif/Bozga/Bouajjani/Habermehl/Moro/Vojnar)
  - ...

Future work

- More general cases of loops.
- Compare/Combine with "Lookahead widening" (CAV 06, Gopan/Reps)
- Implement in NBAC tool [B.Jeannet]